

What is a Continuous Particle Size Distribution?©

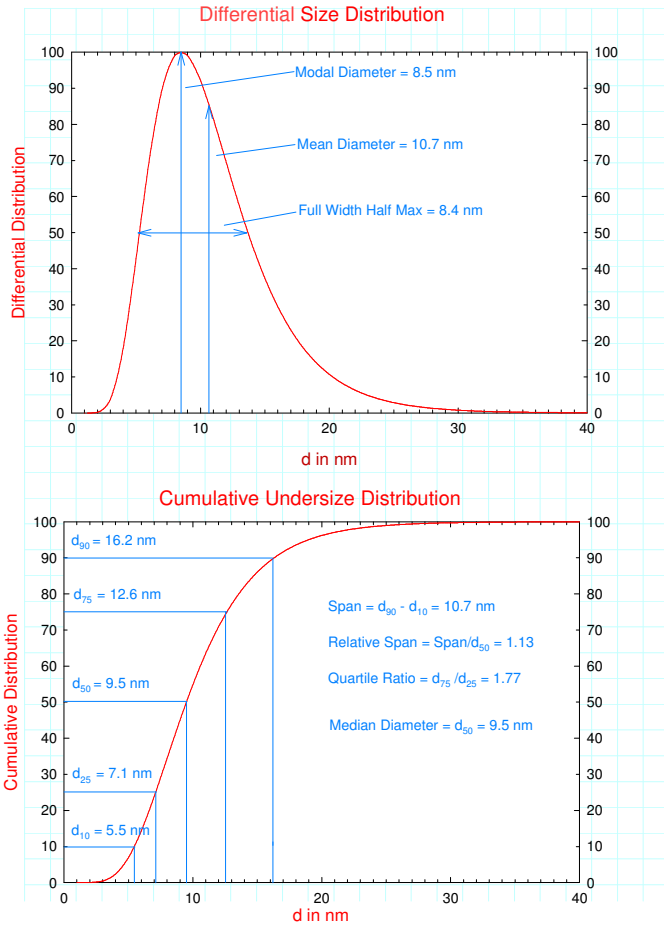
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Particle size distribution data can be presented numerically (tabular format) or graphically. When presented graphically, there are two types: differential and cumulative. They are related. If one differentiates the cumulative distribution curve, the differential distribution is obtained. If one integrates the differential distribution curve, the cumulative distribution is obtained.

Differential Distribution: The differential distribution shows the relative amount* at each size. For example, for the one shown here, using a ruler to draw horizontal and vertical lines, one can determine that the differential amount at 14.5 nm is about 40 while at 18 nm it is about 20. Therefore, the differential distribution is telling us there is twice the amount at 14.5 nm compared to the amount at 18 nm. In addition, the area under the curve between any two diameters divided by the total area under the curve equals the fraction of the total amount between those two diameters.

Measures of central tendency such as the modal and mean diameters are determined from the differential distribution. The modal diameter is the diameter at the peak of the differential distribution. In this example it is 8.5 nm. The mean diameter is the average diameter. In this example it is 10.7 nm.

This distribution is unimodal (single peaked) but not monodisperse (all one size). It has a width. There are several measures of width just as there are several measures of central tendency. One measure of width is FWHM, the full width at half maximum. It is obtained by drawing a horizontal line at 50% of the maximum and taking the difference between the two places it intersects the distribution. In this example it is 8.4 nm.



HHHM, the half width at half maximum, is another measure of width. It is defined as FWHM/2. In this example it is 4.2 nm.

FWHM and HWHM are measures of absolute width. Both have the same units as the diameter, nanometers in this example. A relative fractional measure of width is obtained by dividing either FWHM or HWHM by the measure of central tendency from which it was derived, the modal diameter. In this example the HWHM/Modal Diameter is $4.2/8.5 = 0.49$. A relative percent measure of width is then 49% in this example. Neither measure of relative width has units.

Given that a differential particle size distribution looks like the distribution of repeated measurements of any quantity, it is not surprising that the mathematics of probability distributions permeate the descriptions used in particle size distributions; namely, average (mean), variance, standard deviation, etc. Unfortunately, variance and standard deviation *also* suggest error or uncertainty in any measurement. Indeed, there is uncertainty in particle size distribution measurements; however, when the width of the differential size distribution is represented by the standard deviation of that distribution, it is not meant to suggest it represents the error in the measurement. Rather it is another way of representing the width of the size distribution. In fact, with repeated measurements of the size distribution, one could specify the standard deviation (the error) of the distribution's standard deviation (width of the distribution).

The standard deviation is not shown in the differential distribution above in order to keep the graph readable. If it was, one could define absolute and relative (fractional and percent) standard deviations (dividing by the mean diameter).

The differential distribution shown here is not symmetric about the modal diameter. If it was, and because the diameter axis is linear, then the modal, mean, and median (defined for the cumulative distribution) diameters would all be equal. Such is not the case here. The distribution is skewed to larger sizes. There are various definitions of skew that are derived from probability distributions. In all cases, the skew is positive when the curve tails to the right more than to the left, and it is negative when the curve tails to the left more than to the right. The reference point for tailing is with respect to the modal diameter. A symmetric differential distribution has zero skew.

Cumulative Distribution: The corresponding cumulative distribution is also shown. The cumulative undersize distribution shows the relative amount* at or below a particular size. In this example 50% of the amount of particles is at or below 9.5 nm. Ninety percent are at or below 16.2 nm. These are just two of the possible percentile diameters.

In addition, the cumulative undersize distribution can be used to read the percent between any two diameters. For example, using the example shown above, if $d_{90} = 16.2$ nm and $d_{75} = 12.6$ nm, then it follows that 15% of the amount of particles have sizes between 12.6 and 16.2 nm.

The median diameter is another measure of central tendency. It is the diameter at the 50th percentile, designed d_{50} . Quartile diameters include d_{75} , d_{50} , and d_{25} .

There are several measures of absolute width one can derive given the cumulative distribution. One common measure is the span, $d_{90} - d_{10}$. In this example it is 10.7 nm. A dimensionless measure of width is the relative span defined as span/d_{50} . In this example it is 1.13. Other relative measures of width include percentile ratios such as d_{90}/d_{10} and d_{75}/d_{25} . In this example these values are 2.95 and 1.77, respectively.

The narrower a distribution is the more closely the absolute measures of width approach zero: FWHM, HWHM, variance, standard deviation (square root of variance), and span. Whereas, most of the relative measures of width like d_{90}/d_{10} and d_{75}/d_{25} approach unity.

The following Table summarizes which type of distribution is used to determine various measures of central tendency and measures of the width of the distribution.

Measures of Central Tendency	Distribution Determined From
Mode	Differential
Mean	Differential
Median, d_{50}	Cumulative
Measures of Distribution Width	Distribution Determined From
<i>Absolute</i>	
FWHM	Differential
HWHM	Differential
Standard Deviation	Differential
Span, $(d_{90} - d_{10})$	Cumulative
<i>Relative</i>	
FWHM/Mode	Differential
HWHM/Mode	Differential
Standard Deviation/Mean	Differential
Quartile Ratio d_{75}/d_{25}	Cumulative
d_{90}/d_{10}	Cumulative
$(d_{90} - d_{10})/d_{50}$	Cumulative

Which Distribution Should I Use To Best Present My Data? It depends on what is customary in your particular field. For example, long ago tire manufactures correlated the relative strength of tire treads and tire walls to the quartile ratio d_{75}/d_{25} . And so to represent the distribution width use this relative width obtained from the cumulative distribution.

Suppose you are the first in your field, so there are no guidelines. Then you have the golden opportunity to define which statistics work best. Perhaps you determine that the mean must be (5 ± 1) micron and that 95% must be at or below 20 micron. Thus, d_{95} must be less than or equal to 20 micron. Therefore, you need to show the differential and the cumulative distributions.

Although the examples in this paper are for continuous distributions, in the next paper discrete distributions will be examined. Yet, while we are on the subject of how best to present data, the example of a limited number of size classes in a discrete, cumulative distribution is relevant.

This is the case with sieves where the 4th root of two, 1.189, is the ratio of successive sieve openings. In such a circumstance, while the statistics derived from the cumulative distribution may be useful, trying to numerically differentiate is difficult, leading to large errors in the statistics derived from the resulting differential distribution. In such a case, stay with the statistics derived from the cumulative distribution: the median, span, and percentile ratios.

* The amount could be by number, by surface area, by volume, by mass, by intensity of scattered light. The distributions "weighted" by these quantities are all shifted with respect to one another. The topic of weighting is discussed in another entry in this series on particle sizing.